Chapter-1
CONCEPT OF RELATIONS AND FUNCTIONS

ORDERED PAIR :

An ordered pair consists of two objects or element in a given fixed order.

For example, If A and B are any two sets, then by an ordered pair of elements we mean a pair (a, b) in that order where $a \in A$ and $b \in B$.

EQUALITY OF ORDERED PAIR :

Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff $a_1 = a_2$ and $b_1 = b_2$ i.e. $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

CARTESIAN PRODUCT OF SETS :

Let A and B be any two non empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B is denoted by $A \times B$.

For example If A = {1, 2, 3} and B = {2, 4}

then $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$

RELATION :

Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$. Thus, R is a relation from A to B $\Leftrightarrow R \subseteq A \times B$.

TOTAL NUMBER OF RELATION :

Let A and B be two non-empty finite sets consisting of m and n element respectively. The $A \times B$ consists of mn ordered pairs. So total number of subsets of $A \times B$ is 2^{mn}.

Since each subset of $A \times B$ defines a relation from A to B.

So the total number of relation from A to B is 2^{mn}.

Example : If A = {a, b, c, d}, B = {p, q, r, s} then which of the following are relation from A to B? Give reasons for your answer.

(a) $R_1 = \{(a, p), (b, r), (c, 8)\}$

(b) $R_2 = \{(a, p), (a, q), (d, p), (c, r), (b, r)\}$

(c) $R_3 = \{(q, b), (c, s), (d, r)\}$

Solution : (a) Clearly $R_1 \subseteq A \times B$. So R_1 is relation from A to B.

(b) Clearly $R_2 \subseteq A \times B$. So R_2 is relation from A to B.

(c) Clearly $R_3 \subseteq A \times B \therefore (q, b) \notin A \times B$. So R_3 is not a relation from A to B.

DOMAIN AND RANGE OF A RELATION :

Let R to be a relation from a set A to a set B. Then the set of all final components or cordinates of the ordered pair belonging to R is called the domain, R while the set of all second components or co-ordinate of the ordered pairs in R is called the range of R.

Thus $\quad$ Dom (R) = $\{a : (a, b) \in R\}$

and $\quad$ Range (R) = $\{b : (a, b) \in R\}$

Example : If A = $\{1, 3, 5, 7\}$ and B = $\{2, 4, 6, 8, 10\}$ and let R = $\{(1, 8), (3, 6), (5, 2), (1, 4)\}$ be a relation from A to B. Then find Domain and Range.

Solution: Dom (R) = $\{1, 3, 5\}$

Range (R) + $\{8, 6, 2, 4\}$

RELATION ON A SET :

Let A be a non void set. Then a relation from A to itself *i.e.* a subset of $A \times A$ is called a relation on set A.

INVERSE RELATION :

Let A and B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

clearly, $\quad (a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Dom (R) = Range (R^{-1}) and Range (R) = Dom (R^{-1})

TYPES OF RELATIONS

(i) Void Relation—Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

(ii) Universal Relation—Let A be a set. Then $A \times A \subseteq A \times A$ is called univesal relation on A.

Note—It is to note here that the void and universal relations ona set A are the smallest and largest relation on A respectively.

(iii) Identity Relation—Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words a relation I_A on A is called the identity relation if energy element of A is related to itself only.

(iv) Reflexive Relation—A relation R on a set is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for every $a \in A$

A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Example : Let A = $\{1, 2, 3\}$be a set. Then R = $\{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation. A but R' = $\{(1, 1), (3, 3), (2, 1)\}$ is not a reflexive relation on A because $2 \in A$ but $(2, 2) \in R'$.

Note—1. The identity relation on a non void set A is always reflexive relation on A. How ever a reflexive relation on A is not necessarily the identity relation on A.

2. The universal relation on a non-void set A is reflexive.

3. A relation R on N defined by $(x, y) \in R \Leftrightarrow x \geq y$ is a reflexive relation on N because every natural number is greater than or equal to itself.

(v) Symmetric Relation—A relation R on a set A is said to be a symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.

i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$.

Note—The identity and universal relations on a non void set are symmetric relations.

For example Let A = {1, 2, 3, 4} and let R_1 and R_2 be relations on A given by:

$R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$ and

$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$

Clearly R_1 is a symmetric relation on A.

because $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$

But R_2 is not symmetric relation on A.

because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$

(vi) Transitive Relation : Let A be any set. A relation on R on A is said to be transitive relation iff:

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, \in A$.

i.e. aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

Note—The identity and the universal relation on a non void set are transitive.

For example let A = {1, 2, 3, 4} and let R_1 and R_2 be relation on A given by :

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1)\}$ and

$R_2 = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$

Clearly R_1 is transitive but R_2 is not transitive relation because $(1, 2), (2, 3) \in R_2$ but $(1, 3) \in R_2$.

Example : Check the relation R for reflexivity symmetry and transitivity, where R is defined as $l_1 R l_2$ iff $l_1 \perp l_2$ for all $l_1, h \in A$.

Solution : Let A be the set of all lines in a plane. Given that $l_1 R l_2 \Leftrightarrow l_1 \perp l_2$ for all $l_1, l_2 \in A$.

Reflexivity : R is not reflexive because a line cannot be pependicular to itself *i.e.* $l \perp l$ is not true.

Symmetry : Let $l_1, l_2 \in A$ such that $l_1 \perp l_2$. Then $l_1 \perp l_2 \Rightarrow l_2 \perp l_1$.

So, R is symmetric on A.

Transitivity : R is not transitive because $l_1 \perp l_2$ and $l_2 \perp l_3$ does not implies that $l_1 \perp l_3$.

*** EQUIVALENCE RELATION**

A relation R on a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

Example : Show that the relation R defined on the set A of all triangle in a plane as R = {(T, T_2) : T_1 is similar to T_2} is an equivalence relation.

Solution : Reflexivity : We know that every triangle is similar to itself. Therefore (T, T) $\in$ R for all T$\in$ A $\Rightarrow$ R is reflexive.

Symmetricity : Let (T_1, T_2) $\in$ R Then,

$$\begin{aligned}(T_1, T_2) \in R &\Rightarrow T_1 \text{ is similar to } T_2\\ &\Rightarrow T_2 \text{ is similar to } T_1\\ &\Rightarrow (T_2, T_1) \in R\end{aligned}$$

So, R is symmetric

Transitivity : Let T_1, T_2, T_3 $\in$ A such tha (T_1, T_2) $\in$ R and (T_2, T_3) $\in$ R then (T_1, T_2) $\in$ R and (T_2, T_3) $\in$ R

$\Rightarrow$ T_1 is similar to T_2 and T_2 is similar to T_3

$\Rightarrow$ T_1 is similar to T_3

$\Rightarrow$ (T_1, T_3) $\in$ R

So, R is transitive

Hence, R is an equivalence relation.

Example 2 : Let R be a relation on the set of all lines in a plane defined by (1_1, l_2) $\in$ R $\Leftrightarrow$ l_1 is parallel to l_2. Show that R is an equilence relation.

Solution : Let L be the set of all line in a plane. Then we observe the following properties.

Reflexive : For each line l $\in$ L, we have l || l $\Rightarrow$ (l, l) $\in$ R for every l $\in$ L $\Rightarrow$ R is reflexive.

Symmetric : Let l_1, l_2 $\in$ L such that (l_1, l_2) $\in$ R. Then (l_1, l_2) $\in$ R $\Rightarrow$ l_1 || l_2 $\Rightarrow$ l_2 || l_1

$\Rightarrow$ (l_2, l_1) $\in$ R

So R is symmetric on L.

Transitive : Let l_1, l_2, l_3 $\in$ L such that (l_1, l_2) $\in$ R and (l_2, l_3) $\in$ R. Then

$$\begin{aligned}(l_1, l_2) \in R \text{ and } (l_2, l_3) \in R &\Rightarrow l_1 \parallel l_2 \text{ and } l_2 \parallel l_3\\ &\Rightarrow l_1 \parallel l_3\\ &\Rightarrow (l_1, l_3) \in R\end{aligned}$$

So R is transitive on L.

Hence R is an equivalence relation on L.

FUNCTION :

A relation *f* from a set A to set B is said to be function if every element of set A has one and only one image in set B.

In other words, a function *f* is a relation from a non-empty set A to a non-empty set B such that the domain of *f* is A and no two distinct ordered pairs in *f* have the same first element.

If f is a function from a set A to set B, then we write $f : A \rightarrow B$ which is read as *f* such that A tends to B *f* maps A to B and $(a, b) \in f$. then f (a) = b, where 'b' is called the image of under f and 'a' is pre-image of b under f.

DOMAIN CO-DOMAIN AND RANGE OF A FUNCTION :

Let $f : A \rightarrow B$, Then the set A is known as the domain of f and the set B is known as the co-domain of *f*. The set of all f-images of elements of A is known as the range of *f*.

For example Let A = {–2, –1, 0, 1, 2} and B = {0, 1, 2, 3, 4, 5, 6} consider a rule $f(x) = x^2$. Then f (–2) = 4, f (–1) = 1, f (0) = 0, f (1) =1 and f(2) = 4.

So domain (f) = A = {–2, –1, 0, 1, 2}

and range (f) = {0, 1, 4}

Co-domain (f) = {0, 1, 2, 3, 4, 5, 6}

EQUAL FUNCTION :

Two function f and g are said to be equal iff :

(i) the domain of f = domain of g.

(ii) the co-domain of f = codomain of g.

(iii) f (x) = g(x) for every x belonging to their common domain.

NUMBER OF FUNCTIONS :

Let A and B be two finite sets having m and n elements respectively. Then the total number of function from A to B is n^m.

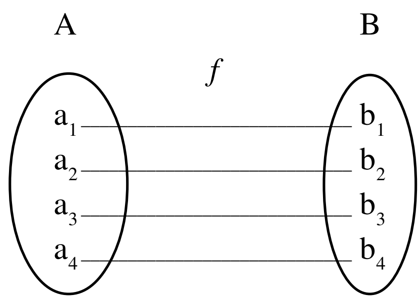
KINDS OF FUNCTIONS

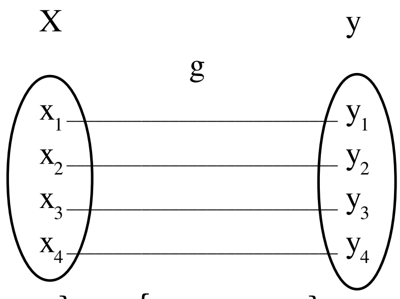
ONE-ONE FUNCTION (INJECTIVE) :

A function $f : A \rightarrow B$ is said to be a one-one function or an injective if different element of A have different image in B.

i.e. for energy $x_1, x_2 \in A$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Let $f : A \rightarrow B$ and $g : x \rightarrow y$ be two functions represented by the following diagram clearly $f : A \rightarrow B$ is a one-one function. But $g : x \rightarrow y$ is not one-one be cause two distinct element x_1 and x_2 have the same image y, under function g.





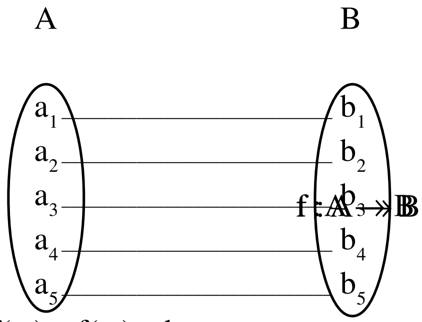
Example : Let $A = \{1, 2, 3, 4\}$; $B = \{1, 2, 3, 4, 5, 6\}$ and a function defined by $f(x) = x + 2$ for all $X \in A$.

Sol. We have {(1, 3), (2, 4), (3, 5), (4, 6)} so f is one-one function.

Many-one Function : A function is said to be many-one function if two or more elements of set A have the same image in B.

i.e. for every $x_1, x_2 \in x$, $f(a_1) = f(a_2) = b_2$ $f(x_1) = f(x_2) \Rightarrow x_2$.

Let $f : A \rightarrow B$ be a function represented by the diagram.



Clearly $a_1 \neq a_2$ but $f(a_1) = f(a_2) = b_2$

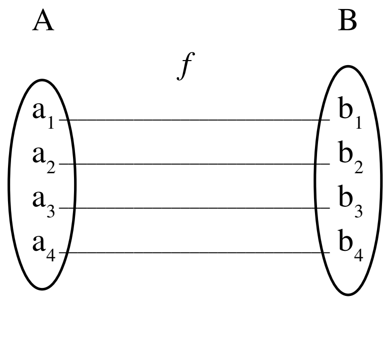
and $a_4 \neq a_5$ but $f(a_4) = f(a_5) = b_4$

So f is many-one function.

ONTO FUNCTION (SUBJECTIVE)

A function $f : A \rightarrow B$ is said to be onto (or Subjective) if every element of is the image of some element of A under of i.e. for every $A \in B$ there exist an element a in A such that $f(a) = b$

Let $f : A \rightarrow B$ be a function represented by the diagram



Example : Let A {–1, 1, 2, –2}, B = {1, 4} and f : A → B be a function defined by f(x) = x^2. Show that *f* is onto.

Sol. We have f(–1) = $(-1)^2$ = 1

f(1) = $(1)^2$ = 1

f(2) = $(2)^2$ = 4

f$(-2)^2$ = $(-2)^2$ = 4

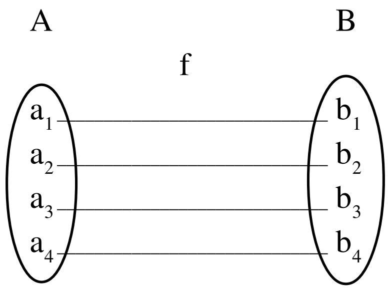
i.e. f(x) = { f(–1), f(–2), f (3), f(1)} = {1, 4} = B

so f is onto.

ONE-ONE AND ON TO FUNCTION (BIJECTIVE) :

A function f : A → B is said to be bijective it is one-one as well as onto.

Let f : A → B a function represented by the following diagram.



Example : Let A = R – {2} and B = R – {1}. If f : A → B is a function defined by $f(x) = \frac{x-1}{x-2}$ show that f is bijective.

Sol. One-one function : Let x, y be any two elements of then

f(x) = f(y)

⇒

⇒ (x – 1) (y – 2) = (x – 2) (y – 1)

⇒ xy – y – 2x + 2 = xy – x – 2y + 2

⇒ x = y

Thus f (x) = f(y) ⇒ x = y for all x, y ∈ A so f is one-one function.

ONTO-FUNCTION :

Let y be an arbitrary element of B then

f(x) = y

⇒

$\Rightarrow \quad (x - 1) = y\,(x - 2)$

$\Rightarrow \quad x = \dfrac{1-2y}{1-y}$

Clearly is a real number for all $y \neq 1$

So f(x) =

So f is onto function.

Hence f is bijective.

COMPOSITION OF FUNCTIONS :

Let $f : A \to B$ and $g : B \to C$ be two functions. Then a function g of $f : A \to C$ defined by

$$g \text{ of } (x) = g\,(f(x)), \text{ for all } x \in A$$

is called the composition of f and g.

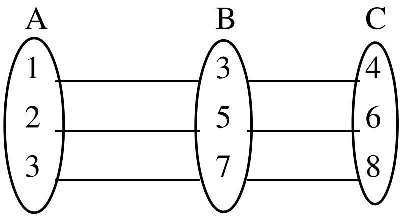
Consider the two functions given below

Let $f : A \to B$ and $g : B \to C$

Now $\quad f(x) = 2x + 1, x \in \{1, 2, 3\}$

and $\quad g\,(y) = y + 1, y \in \{3, 5, 7\}$

$$f\left(\frac{1-2y}{1-y}\right) = \frac{\frac{1-2y}{1-y}-1}{\frac{1-2y}{1-y}-2} = y$$



Example : If the function $f : R \to R$ be given by $f(x) = x^2 + 2$ and $g : R \to R$ be given by $g(x) = \dfrac{x}{x-1}$ find fog and gof.

Sol. $fog(x) = f\,(g(x)) = (g(x))^2 + 2$

$$= \frac{x^2}{(x-1)^2} + 2$$

and gof(x) = g(f(x)) = $\frac{f(x)}{f(x)-1} = \frac{x^2+2}{x^2+2-1}$

$=$

INVERSE OF A FUNCTION :

A function f : x $\rightarrow$ y is defined to be invertible, if there exists a function g : y $\rightarrow$ x such that gof = I_x and fog = I_y The function g is called the inverse of f and is denoted by f^{-1}.

If f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible. This fact significantly helps for proving a function of to be invertible by showing that f is one-one and onto specially when the actual inverse of f is not to be determined.

Example : Let f : N $\rightarrow$ y be a function defined as f(x) = 4x + 3, where y = { y $\in$ N : y = 4x + 3 for some x $\in$ N}. Show that f is invertible find the inverse.

Sol. Consider an arbitrary element y of Y. By the definition of Y, y = 4x + 3 for some x in the domain N. This shows that

$x = \frac{y-3}{4}$

$\frac{x^2+2}{x^2+1}$

$\frac{4(y-3)}{4} + 3$

i.e. $g(y) = \frac{y-3}{4}$ define g : y $\rightarrow$ N

Now go f(x) = g (f(x)) = $\frac{f(x)-3}{4}$

$= \frac{4x+3-3}{4}$

gof f(x) = x = I_n

and fog (y) = f(g(y)) = 4g(y) + 3

$=$

= y – 3 + 3

= y

fog (y) = y = I_Y

so, f is invertible and g is the inverse of f.

BINARY OPERATIONS :

If A and B be two non empty sets, then a function from A × A to A is called a binary operation on A. It is denoted by '*' the unique element of A associated with the pair (a, b) of A × A is denoted by a * b.

PROPERTIES OF BINARY OPERATIONS :

1. Binary operation is commutative
   i.e. a * b = b * a for all a, b ∈ A
2. Binary operation is associative
   i.e. (a * b) * c = a * (b*c) for all a, b, c ∈ a}
3. An element e ∈ A is said to be an identity element iff e * a = a = a * e
4. An element a ∈ A is called invertible iff there exists some b ∈ A such that a * b = b * a = e, b is called inverse of A.

MISCELLANEOUS QUESTIONS

Part A

1. Show that the relation 'is a fator of' from R to R is reflexive and transitive but not symmetric
2. Show that each of the relation R in the set $f(x) = |x|\ \forall x \in R$, $f(x) = 115x + 49\ \forall x \in N$, $f(x) = \frac{x^2}{4} + 2,\ g(x) = 1 - \frac{1}{1-x},\ x \neq 1$
   $A = \{x \in z : 0 \leq x \leq 12\}$ given by
   (a) R = {(a, b) : |a – b| is multiple of 4}
   (b) R = {(a, b) : a = b} is a equivalence relation.
3. Let R be a relation on the set of all lines in a plane defined by $(l_1, l_2) \in R \Rightarrow$ line l_i is parallel to l_2. Show that R is an equivalence relation.
4. Which of the following functions are onto function if $f : R \to R$.
   (a)
   (b)
5. Which of the following functions are many-to-one function?
   (a) $f : \{-2, -1, 1, 2\} \to \{2, 5\}$ defined as $f(x) = x + 1$
   (b) $f : \{0, 1, 2\} \to \{1\}$ defined as $f(x) = 1$
   (c) $f : N \to N$ defined as $f(x) = 5x + 7$
6. Find fog, goft, fof and gog for the following functions :

7. Let f (x) = | x |, g (x) = [x]. Verify that fog $\neq$ gof.

8. Find the Inverse of each of the following function (if it exist).

(a) $f(x) = x + 3 \quad \forall x \in R$

(b)

(c)

9. If A = {1, 2}. Find the total nubmer of binary operations on A.

10. Let * be the binary operation defined on Q by for all a, b, $\in$ Q^+, then find the inverse of 4*6.

11. A binary operation * on Q – {–1} is defined by a*b = a + b + ab for all a, b $\in$ Q – {–1}. Find identity element on Q. Also find the inverse of an element in Q – {–1}.

MISCELLANEOUS QUESTIONS

Part-B

1. Write for each of the following functions gof, gof, fof, gog.

(a) $f(x) = x^3, \; g(x) = 4x - 1$

(b) $f(x) = \sqrt{x - 4} \;\; x \geq 4, \; g(x) = x - 4$

(c) [illegible] $= x^2 + 1$ 34

2. Without using graph prove that the function f : R $\rightarrow$ R defined f (x) = 4 + 3x is one-to-one.

3. Prove that f : R $\rightarrow$ R defined by f (x) = $4x^3 - 5$ is a bijection.

4. Which of the following equations describe a function whose inverse exists :

(a)

(b)

(c)

(d) $f(x) = \dfrac{3x + 1}{x - 1}$

5. If gof (x) = | sin x | and gof (x) = then find f (x) and g (x).

6. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} ad R = {(a, b) : b = a + 1} is reflexive, symmetric and transitive.

7. Let * be a binary operation on Q defined by for all, a, b $\in$ Q. Prove that * is commutative on Q.

8. Let * be a binary operation on the set Q of rational numbers define by $a * b = \frac{ab}{5}$ for all a, b $\in$ Q, show that * is associative on Q.

$$a * b = \frac{a+b}{3}$$

Chapter-2

CONCEPT OF INVERSE TRIGONOMETRIC FUNTIONS

A function f : A $\rightarrow$ B is invertible iff it is a bijection. The inverse of f is denoted by f^{-1} and is defined as

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

Clearly, domain of f^{-1} = range of f and range of f^{-1} = domain of f.

Consider the sine function with domain R and range [–1, 1] clearly is not invertible. If the restrict the domain of it in such a way that it become one-one, then it would be come invertible. If we consider sine as a function with domain $[-\pi/2, \pi/2]$ and $x \in [-1, 1]$ $\sin^{-1}x$ has infinitely many values for given $x \in [-1, 1]$. $\sin^{-1}x$ has infinitely many values for given $x \in$ [–1–, 1]. There is one value among these values which lies in the interval . This value is called the principal value.

DOMAIN AND RANGES OF INVERSE TRIGONOMETRICAL FUNCTIONS

Function	Domain	Range
$\sin^{1}x$	[–1, 1]	[–π/2, π/2]
$\cos^{-1}x$	[–1, 1]	[0, π]
$\tan^{-1}x$	(–∞, ∞)	(–π/2, π/2)
$\cot^{-1}x$		(–∞, ∞) (0, π)
$\sec^{-1}x$	(–∞, –1) ∪ (1, ∞)	[0, π/2)∪ (π/2, π]
$\text{cosec}^{-1}x$	(–∞, –1) ∪ (1, ∞)	

$[-\pi/2, \pi/2]$ $\left(\frac{1}{\sqrt{2}}\right)$ $\theta = \frac{\pi}{4}$ $0 = \theta$ $\left[0, \frac{\pi}{2}\right]$

Example : Find the principal value of $\sin^{-1}$

Sol. Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$

Or $\quad \sin\theta = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$

Or

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS :

1.

2.

3.

4.

5.

6.

7. (i) [illegible]

(ii) $\cos^{-1}(-x) = \pi = \cos^{-1} x, x \in [-1,1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$

(iv)

(v)

(vi)

8. (i)

(ii) $\tan^{-1} x + \cot^{-} x = \frac{\pi}{2}, x \in R$

(iii)

9. (i)

$$= \sec^{-}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) $$\cos^{-1} = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

(iii)

$$= \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) \quad \frac{\sqrt{1-x^2}}{x} = \cot^{-1}\left(\frac{1}{x}\right)$$

10. (i)

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \pi \text{ if } xy > 1$$

(ii) $$\tan^{-x} - \tan^{-1} = \tan^{-1}\left(\frac{x-y}{1+xy}\right), xy > -1$$

(iii) $$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$$

if $x, y \geq 0, x^2 + y^2 \leq 1$

if $x, y \geq 0, x^2 + y^2 > 1$

(iv)

if $x, y \geq 0$ and $x^2 + y^2 \leq 1$

11. (i)

(ii) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$

(iii)

12. (i)

(ii) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

(iii) $3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ [illegible]

13.

Example : Solve the equation

Sol. Let $x = \tan\theta$, then

$$\tan^{-1}\left(\frac{1 - \tan\theta}{1 + \tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$$

$$\Rightarrow \quad \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta\right)\right] = \frac{1}{2}\theta$$

$$\Rightarrow \quad \frac{\pi}{4} - \theta = \frac{\theta}{2}$$

$\Rightarrow$

$\therefore$

Example : Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), \frac{-\pi}{2} < x < \frac{3\pi}{2}$ in the simplest form.

Sol. $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$$

MISCELLANEOUS QUESTIONS

Part-A

1. Find the principal value of each of the following :

   (a)

   (b) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

   (c)

   (d) $\cot^{-1}(1)$

   (e)

   (f)

2. Evaluate :

   (a) $\cos\left(\cos^{-1}\frac{1}{3}\right)$

   [illegible]

   (b)

   (c)

   (d)

   (e) $\text{cosec}[\cot^{-1}(-\sqrt{3})]$

3. Simplify :

   (a) $\tan\left(\text{cosec}^{-1}\frac{x}{2}\right)$

   (b) $\sec(\tan^{-1}x)$

   (c)

(d) $\cos(\cot^{-1} x^2)$

(e) $\cot(\text{cosec}^{-1} x^2)$

4. Solve the equation $\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}(3x)$.

5. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then

   Prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

6. Prove each of the following :

   (a)

   (b)

   (c)

7. If $\cos^{-1}(x) + \cos^{-1} y = B$, prove that $x^2 - 2xy \cos\beta + y^2 = \sin^2\beta$.

8. Evaluate each of the following :

   (a) [illegible]

   (b) $\cot(\tan^{-1} c + \cot^{-1} c)$

   (c)

   (d)

MISCELLANEOUS QUESTION

Part-B

1. If $f : R \rightarrow R$ defined by $f(x) = x^3 + 4$. What will be f^{-1}.

2. Solve the equation :

3. Show that :

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

4. Prove that :

(a) $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

(b)

(c)

(d) $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=\frac{\pi}{4}-x$

(e) $\cos^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ac+1}{c-a}\right)=0$

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$$

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}$$

Chapter - 3
MATRICES AND DETERMINANTS

MATRIX :

A set of mn numbers (real or imaginary) arranged in the form of a rectangular array of m rows and n columns is called $m \times n$ matrix.

An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{ij} & \dots & a_{1n} \\ a_{21} & a_{22} & \vdots & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & a_{23} & \dots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & \vdots & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & a_{i3} & \dots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \vdots & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

In Compact form the above matrix is represented by A - $[a_{ij}]_{m \times n}$ or $A = [a_{ij}]$

TYPES OF MATRICES :

(i) Row Matrix : A matrix having only one row is called a row-matrix.

For example, $A = [12 \quad -15 \quad -2]$ is a row matrix of order 1×4

$$\begin{bmatrix} 11 & 02 & 03 \\ 01 & 14 & 01 \\ 02 & 03 & 17 \end{bmatrix}$$

(ii) Column matrix : A matrix having only one column is called a column matrix.

For Example, $\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$ is a column matrix of order 4×1

(iii) Sqaure Matrix : A matrix in which the number of rows is equal to the number of column, say n is called a square matrix of order n.

For Example, the matrix is sqaure matrix of order 3 in which the diagonal elements are 1, 4 and 7.

(iv) Diagonal Matrix : A square matrix $A = [a_{ij}]_{mn}$ is called a diagonal matrix if all the elements except those in the leading diagonal are i.e. $a_{ij} = 0$ for all $i \neq j$

For Example The matrix is a diagonal matrix and it is denoted by diag [1, 2, 3]

(v) Scalar Matrix : A sqaure matrix $A = [a_{ij}]$ $i \neq j$ and a_{ij} = constant for all i.

In other word, a diagonal matrix in which all the diagonal elements are equal is called the Scalar matrix.

For Example : The matrix are Scalar matrix.

(vi) Identity Or Unit Matrix : A squre matrix $A = [a_{ij}]_{n \times n}$ is called an identity or unit matrix if $a_{ij} = 0$ for all $i \neq j$ amd $a_{ij} = 1$ for all i.

In other words, a diagonal matrix in which all the diagonal elementsare equal to 1 is called the matrix, and it is denoted by I.

For Example

The matrices are null matices of order 2×2 and 3×3 respectively.

(vii) Null Matrix : A matrix whose all elements are zero is called a null matrix or a zero matrix.

For Example are upper triangular matrix

(ix) Lower Trianguloar Matrix : A sqaure matrix $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$ for example A = is a lower triangular matrix of order 3.

EQUALITY OF MATRICES :

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{r \times s}$ are equal if,

(i) m = r i.e. the number of rows in of equals the number of rows in B.

(ii) n = s i.e. the number of column in of equals the number of column in B.

(iii) $a_{ij} = b_{ij}$ for i = 1, 2, ... m and j = 1, 2, 3....n.

If two matrices A and B are equal, we write A = B otherwise we write $A \neq B$.

Example : Find the values of x, y, z and a which satisfy the matrix equation

Sol. Since the corresponding elements of two equal matrices are qual.

$$\therefore\ x + 3 = 0,\ 2y + c = -7,\ z - 1 = 3 \text{ and } \underline{4a - 6\ 6 = 2a}$$

Solving these we get

$$a = 3,\ x = -3,\ y = -2,\ z = 4$$

ADDITION OF MATRICES :

The sum of two matrices is defined only when they are of the same order.

If A and B be two matrices, each of order $m \times n$. Then their sum A + B is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Example : If then find A + B.

Sol.

PROPERTIES OF MATRIX ADDITION :

(i) Matrix addition is commutative i.e. If A and B are two $m \times n$ matrices, then

$A + B = B + A$

(ii) Matrix addition is associative i.e. If A, B, < are three matrices of the same order then $(A+B) + C = A + (B + C)$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \quad A + B = \begin{bmatrix} 5+1 & 2+2 & 7+3 \\ 1+2 & 4+1 & 5+4 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 10 \\ 3 & 5 & 9 \end{bmatrix}$$

(iii) Existence of Indentity : The null matrix is the identity element for matrix addition i.e. $A + O = A = O + A$

(iv) Existence of Inverse : For every matrix $A = [a_{ij}]_{m \times n}$ these exists a matrix $[-a_{ij}]_{m \times n}$ denoted by $-A$ such that

$A + (-A) = O = (-A) + A$

(v) Cancellation laws hald good in case of addition matrices.

If A, B, C are matrices of same order then

$A + B = A + C \Rightarrow B = C$ and $B + A = C + A \Rightarrow B = C$

SUBTRACTION OF MATRICES :

If A and B are two matrices of same order the define

$$A - B = A + (-B)$$

MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be any number called a scalar, then scalar multiplication is defined as

$KA = [K a_{ij}]_{m \times n}$

For example : If A then

MULTIPLICATION OF MATICES :

The Product two matrices A and B is defined if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]_{m \times n}$ and

$B = [b_{ij}]_{n \times p}$ are two matrices then the Product AB is the matrix C of order $m \times p$

For example : If

Then product AB is defined and is given by . This is a 2×2 matrix.

and $B = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}_{3 \times 2}$

PROPERTIES OF MATRIX MULTIPLICATION :

(i) Matrix multiplication is not commutative in general i.e. $AB \neq BA$

(ii) Matrix multiplication is associative i.e.

$(AB)\,C = A(BC)$

(iii) Matrix multiplication is distributive over matrix addition

i.e. $A\,(B + C) = AB + AC$

(iv) If A is an $m \times n$ matrix then

$I_m.A = A = A.I_n$

TRANSPOSE OF A MATRIX :

If $A = [a_{ij}]$ be an $m \times n$ matrix then the transpose of denoted by A^T or A' is an $n \times n$ matrix.

Thus A^T is obtained from A by changing its rows into columns and its columns into rows.

For example : If , then

PROPERTIES OF TRANSPOSE OF MATRIX

If A and B be two matrices then,

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$, A and B being of the same order.

(iii) $(KA)^T = KA^T$, K be any Scalar

(iv) $(AB)^T = B^T A^T$

SYMMETRIC AND SKEW-SYMMETRIC MATRICES

SYMMETRIC MATRIX :

A square matrix of is called a Symmetrix if $A^T = A$

i.e. $a_{ij} = a_{ji}$ for all i, j

SKEW—SYMMETRIC MATRIX :

A square matrix A is called a Skew-Symmetric matrix if

$A^T = -A$

i.e. $a_{iJ} = -a_{ji}$ for all i, j.

Note : 1. If A be a square matrix then,

(i) $A + A^T$ is a symmetrix matrix

(ii) $A - A^T$ is a Skew-Symmetric matrix

(iii) $A.A^T$ and $A^T.A$ are Symmetric matrix

2. Every square matrix can be uniquely expressed as a sum of a symmetric matrix and a skew symmetric matrix.

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 4 & 1 \\ 4 & 1 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2j} & \ldots\ldots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i12} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{in} \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nj} & \ldots\ldots & a_{nn} \end{vmatrix}$$

DETERMINANTS :

Every square matrix can be associated to an expression or a number which is known as its determination. If $A = [a_{ij}]$ is a square matrix of order n, then determinant of A is denated by |A|.

i.e.

DETERMINANT OF A SQUARE MATRIX OF ORDER 1 :

If A = [a_{11}] is square matrix of order 1; then determinant of A is defined as |A| = A_{11}

DETERMINANT OF A SQUARE MATRIX OF ORDER 2 :

If is a square matrix of order 2, then determinant of A is defined as

DETERMINANT OF A SQUARE MATRIX OF ORDER 3 :

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3 then determinant of A is defined as

$$|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} \\ a_{31} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31} - a_{31}a_{32})$$

SINGULAR MATRIX :

A square matrix is a singular matrix if its determinant is zero.

i.e. |A| = 0

NON-SINGULAR MATRIX :

A square matrix is a non-singular matrix if its determinant is not zero.

i.e. $|A| \neq 0$

ADJOINT OF A SQUARE MATRIX :

Let A = [a_{ij}] be a square matrix of order n and let C_{ij} be co factor of a_{ij} in of. Then the transpose of the matrix of cofactor of elements of A is called the adjoint of A and it is denoted by 'adj A'.

Thus adjA [c_{ij}]^T

Where c_{ij} denotes the cofactor of a_{ij} in A.

Note—Let A be a square matrix of order n. Then A (adjA) = |A| In = (adjA) A

INVERSE OF A MATRIX :

A square matrix of order n is invertible if there exists a square matrix 8 of same order such that

$$AB = I_n = BA$$

where I_n is identity matrix of order n.

Note— 1. A square matrix is invertible iff it is non-singular.

2. Every invertible matrix possesses a unique matrix.

The inverse of A is given by,

$$A^{-1} = \frac{1}{|A|}.\text{adj}A$$

ELEMENTARY TRANSFORMATIONS OR ELEMENTARY OPERATIONS OF A MATRIX

(i) Intercharge of any two rows (or columns) : if i^{th} row (column of a matrix is inter changed with j^{th} row (column), it will be denoted by,

$$R_1 \leftrightarrow R_j (ci \leftrightarrow cj)$$

(ii) Multiplying all elements of a row (column) of a matrix by a non zero Scalar : If the elements of i^{th} row (column) are multipied by a non zero scalar K, it will be denoted by $R_i \to kR_i [ci \to cj]$

(iii) Adding to the elements of a row (coloum) the corresponding elements of any other

(iv) Row (column) multiplied by any Scalar K : If K times of elements of j^{th} row (column) are added to the corresponding elements of the i^{th} row (column) it will be denoted by,

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$Ri \to R + KRj \; (ci \to ci + Kcj)$$

Example : Find the inverse of the matrix of where,

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

Sol. We have,

$$A = IA$$

Or

Applying $R_1 \to R_1 - R_2$

$\Rightarrow$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$\Rightarrow$

Applying

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1/2 \\ 0 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2$ and
$R_3 \to R_3 + 2R_2$

$\Rightarrow$

Applying [illegible]

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5/2 & 3/2 & 1/4 \end{bmatrix}$$

Applying

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A$$

Hence

PROPERTIES OF DETERMINANTS :

1. The value of a determinant remains unchanged if its rows and columns are interchanged.
2. If two rows (or column) of a determinant are interchanged then the value of the determined changes in sign only.
3. If any two rows (or columns) of a determinant are identical the value of the determinant is zero.
4. If each element of a row (or column) of a determinant is multiplied by the same constant say, K $\neq$ O then the value of the determinant is multiplied by that Constant K.
5. If each element of a row (or of a column) of a determinant is expressed as the sum (or difference) of two or more terms then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order whose remaining rows (or columns) do not change.
6. The value of determinant does not change, if to each element of a row (or a column) be added (or subtracted) the some multiples of the corresponding element of one or more other rows (or columns).

APPLICATIONS OF DETERMINANTS

1. AREA OF TRIANGLE :

If A(x_1, y_1), B (x_2, y_2) and C(x_3, y_3) are the vertices of ΔABC then

Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

2. CONDITION OF COLLINEARITY OF THREE POINTS :

If A (x_1, y_1), B(x_1, y_2) and c(x_3, y_3) be three points then A, B, C are collinear if area of $\Delta ABC = 0$

i.e. $\Delta ABC = 0 \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

3. Equation of line passing through two points A(x_1, y_1) and B(x_2, y_2) is

SOLUTION OF A SYSTEM OF LINEAR EQUATION BY MATRIX METHOD

In this method, we first Express the given system of equation in the matrix form Ax = B whose A is called the coefficient matrix.

For Example, if the Given system of equation is $a_1x + b_1y + 4z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$, then this system is expressed in the matrix equation form as

Where A =

and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Note.—If A is singular, then |A| = 0. Hence A^{-1} does not exist and so this method does not work. This method work only when A is non singular.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

CRITERION FOR CONSISTENCY OF A SYSTEM OF EQUATIONS :

If AX = B be a System of two or three linear equations then we have :

(i) If $|A| \neq 0$, then the system of equation is consistent has unique solution, given by $X = A^{-1}B$.

(ii) If |A| = 0 and (adjA)B = 0 then the system is consistent and has infinitely many solutions.

(iii) if |A| = 0 and (adjA) $B \neq 0$ then the system is inconsistent and the system of equation have no solution.

MISCELLANEOUS QUSTION

Part-A

1. Construct a 3×2 matrix whose elements in the ith row and Jth column is given by :

   (a) $i + 3J$

   (b) 5iJ

   (c) $i + J - 2$

   (d) **j^J**

2. Find the values of a, b, c and d if :

(a)

(b)

(c)

3. Can a matrix of order 1×2 be equal to a matrix of 2×1 ?

4. If then find (-7) A.

5. Find A^T (transpase of A) :

(a)

[illegible]

(b)

(c)

6. Show that the given matrices are symmetric matrix.

(a)

(b)

(c)

(d)

7. Show that each of the following matrices is a skew-symmetric matrix :

(a)

(b)

(c) [illegible]

(d)

8. If and then find :

(a) A + B

(b) A – B

(c) – A + B

(d) 3A + 2B

(e) Ab

(f) $A^T B^T$

9. If and then find :

(a) A^T

(b) $(A + B)^T$

(c) $A^T + B^T$

10. If and Find AB and BA. Is AB = BA.

11. Find the values of x and y, if

(a) [illegible]

(b)

12. Find the inverse of the following matrices using elementary operations :

(a)

(b)

(c)

MISCELLANEOUS QUESTION

Part-B

1. Construct a matrix of order 3×2 whose elements a_{iJ} are given by :

   (a) $a_{iJ} = \underline{d\ i - 2j}$

   (b) $a_{iJ} = 3i - J$

   (c)

2. Find the value of x, y and z if :

   (a) $\begin{bmatrix} x+y & z \\ 6 & x-y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 6 & 4 \end{bmatrix}$

   (b)

3. Find X, if :

   (a) [illegible] $+ X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   (b)

4. Find A (B + C), if and

5. Find the inverse of :

   (a)

   (b)

   (c)

(d)

(e)

MISCELLANEOUS QUESTIONS

Part-C

1. Find $|A|$, if :

(a)

(b)

(c) [illegible]

(d)

2. Find which of the following matrices are singular matrices :

(a)

(b)

(c)

(d)

3. Expand the determinant by using row :

(a)

(b)

(c) [illegible]

(d)

4. Find the minors and cofactors of the elements of the second row and third column of the determinant?

(a)

(b) $\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

5. Solve for x the following equation :

(a)

(b)

(c)

(d)

[illegible] $= 27(x+1)$

6. Show that :

(a)

(b) $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

(c) $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1+b & 1+c \end{vmatrix} = bc+ca+ab+abc$$

7. $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ gives what ?

8. $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ gives what ?

9. Find the area of ΔABC when A, B and C are (3, 8), (4, – 2) and (5, –1) respectively.

MISCELLANEOUS QUESTION

Part-D

1. Find all the minors and cofactors of :

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

2. Evaluate [illegible] by expanding.

[illegible]

3. Solve for (x), if

4. Using property of determinant, show that :

(a)

(b)

5. Evaluate :

(a) $$\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$$

(b) $$\begin{vmatrix} 1 & \omega^3 & \omega^5 \\ \omega^3 & 1 & \omega^4 \\ \omega^5 & \omega^5 & 1 \end{vmatrix}$$

6. Using determination find the valueof K so that the following points become collinear.

(1) (k, 2 – 2k), (– k + 1, 2k) and (–4 – k, 6 – 2k)

(2) (k, –2), (5, 2) and (6, 8)

(3) (3, –2), (5, 2) and (8, 8)

Chapter - 4
CONCEPT OF LIMITS AND CONTINUITY

LIMITS :

In General as $x \to a$, $f(x) \to l$ then l is called limit of the function f(x) which is symbolically written as $\lim_{x \to a} f(x) = l$.

LEFT HAND LIMIT :

Left hand limit of a function f(x) is that of f(x) which is dictated by the values f(x) when x tends to a from the left.

We say given the values of f near x to the left of a. This value is called the left hand limit of f at .

RIGHT HAND LIMIT :

Right hand limit of a function f(x) is that of f(x) which is dictated by the values of f(x) when x tends to a from the right we say $\lim_{x \to a^+} + f(x)$ given the values f near x to the right of a. This value is called the right hand limit of f(x) at a.

If the right and left hand limits coincide, we call that $\lim_{x \to a} f(x)$ $\lim_{h \to 0+} f(a+h)$ common value as limit of f(x) at x = a and denote it by $\lim_{x \to a} f(x)$.

EVALUATION OF LEFT HAND LIMITS :

To evaluate L.H.L. of f(x) at x = a i.e. $\lim_{x \to \bar{a}} f(x)$ we proceeds following :

Step I : Write $\lim_{x \to \bar{a}} f(x)$

Step II : Put x = a – h and replace $x \to \bar{a}$ by $h \to a$ to obtain

Step III : Simlify $\lim_{h \to 0} f(a-h)$ by usin the formual for the given function.

Step IV : The value obtained in Step III is the LHL of f(x) at x = a

EVALUATION OF RIGHT HAND LIMITES :

To evaluate RHL of f(x) at x = 0 i.e. $\lim_{x \to a} + f(x)$ we proceed as follows "

Step I : Write the

Step II : Put x = a + h and replace $x \to a^+$ by $h \to O$ to obtain .

Step III : Simplify by using the formula for the given function.

Step IV : The value obtained in Step III is the RHL of f(x) at x = 0

Example : Evaluate the Left hand and Right hand limits of the function

:

Sol. L.H.L. = $\lim_{x\to 4} f(x)$

$= \lim_{h\to o} f(4-h)$

$= \lim_{h\to o} \frac{(14-h-41)}{4-h-4} = \lim_{h\to o} \frac{|-h|}{-h}$

$= \lim_{h\to o} \frac{4}{-h}$

$= -1$

$\lim_{h\to 0} f(a+h)$ $\quad \lim_{x\to a} k f(x) = k \lim_{x\to a} f(x)$ $\quad f(x) = \left\{ \frac{|x-4|}{x-4}, x \neq x, o, x = 4 \text{ at } x = 4 \right\}$

R.H.L. $= \lim_{x\to 4+} f(x)$

$= \lim_{h\to 0} f(4+h)$

$= \lim_{h\to 0} \frac{14+h-41}{4+h-4}$

$= 1$

THE ALGEBRA OF LIMITS

1. $\lim_{x\to a} k = k$

2.

3. $\lim_{x\to a} k.\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

4. $\lim\limits_{x \to a} f(x).g(x) = \lim\limits_{x \to a} f(x).\lim\limits_{x \to a} g(x)$

5. $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$

6. $\lim\limits_{x \to a} f(x)^{g(x)} = \lim\limits_{x \to a} f(x)^{\lim\limits_{x \to a} g(x)}$

7. $\lim\limits_{x \to a} e^{f(x)} = e^{\lim\limits_{x \to a} f(x)}$

EVALUATION OF LIMITS

1. DIRECT SUBSTITUTION METHOD :

If by direct substitution of the point in the given expression we get a finite number the number obtained is the limit of the given expression.

For Example : 1. $\lim\limits_{x \to 2}(2x^2 + 3x + 5)$

Sol. $\lim\limits_{x \to 2}(2x^2 + 3x + 5)$

$= 2(2)^2 + 3 \times 2 + 5)$

$= 8 + 6 + 5$

$= 19$

2. FACTORISATION METHOD :

Consider the limit $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$. If by putting x = a the rational function $\frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ etc. them (x – a) is a factor of both f(x) and g(x). In such case use factorise the numerator and denominator and them cancel out the common factor (x– a). After cancelling out common factor we put x = a and obtained the value.

For example : $\lim\limits_{x \to 2} \frac{x^3 - 2^3}{x^2 - 2^2}$

Sol. $\lim\limits_{x \to 2} \frac{x^3 - 2^3}{x^2 - 2^2} = \frac{(x-2)(x^2 + 4 + 2x)}{(x-2)(x+2)}$

$$= \lim_{x \to 2} \frac{(x^2 + 4 + 2x)}{(x + 2)}$$

$$= \frac{4 + 4 + 4}{2 + 2}$$

= 3

3. RATIONALISATION METHOD :

Example : 1.

Sol. $\lim_{x \to a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3ax} + 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}}$

$$= \lim_{x \to a} \frac{(a-x)\sqrt{3a+x} + 2\sqrt{x}}{3(a-x)\sqrt{a+2x} + \sqrt{3x}}$$

$$\lim_{x \to a} \frac{12\sqrt{a+2x} - \sqrt{3x}}{4\sqrt{3a+x} - 2\sqrt{x}}$$

$$= \lim_{x \to a} \frac{\left(\sqrt{3a+x} + 2\sqrt{x}\right)}{\left(3\sqrt{a+2x} + \sqrt{3x}\right)}$$

$$= \frac{4\sqrt{a}}{2\left(3\sqrt{3a}\right)}$$

$$= \frac{2}{3\sqrt{3}}$$

EVALUTION OF LIMITS BY US IN STANDS RESULTS :

1. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

2. $\lim_{x \to a} \frac{x^m - a^n}{x - a} = \frac{m}{n} a^{m-n}$

3. $\lim_{x\to 0}\frac{\sin x}{x}=1$

4. $\lim_{x\to 0}\frac{\tan x}{x}=1$

5. $\lim_{x\to 0}\frac{\sin^{-1} x}{x}=1$

6. $\lim_{x\to 0}\frac{\tan^{-1} x}{x}=1$

7. $\lim_{x\to 0}\frac{\sin x^{0}}{x}=\frac{\pi}{180}$

8. $\lim_{x\to 0}\cos x=1$

9. $\lim_{x\to a}\frac{\sin(x-a)}{(x-a)}=1$

10. $\lim_{x\to a}\frac{\tan(x-a)}{x-a}=1$

$\lim_{x\to a} f(x)=f(x) \Leftrightarrow \lim_{x\to \bar{a}} f(x)=f(x)=f(a)$

11. $\lim_{x\to 0}\frac{a^{x}-1}{x}=\log c^{a}$

12. $\lim_{x\to 0}\frac{e^{x}-1}{x}=1$

13. $\lim_{x\to 0}\frac{e^{x}-b^{x}}{x}=\log e^{\left(\frac{a}{b}\right)}$

14. $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

CONTINUITY AT A POINT :

A function f(x) is said to be continous at a point x = a of its domain iff $\lim_{x\to a} f(x)=f(x)$

i.e.

MISCELLANEOUS QUESTIONS

Part-A

1. Evaluate each of following limits :

   (a)

   (b)

   (c)

   (d)

   (e)

   (f)

   (g) $\lim \; \ldots \begin{cases} \ldots, & x \le 1 \\ \ldots, & x > 1 \end{cases}$ as $x \to 1$

   (h)

   (i)

   (j)

2. Show that does not exist.

3. Find the left hand limit and right hand limits of the functions.

   (a)

(b) If $f(x)=\begin{cases} x^2 & , x \le 1 \\ 1 & , x > 1 \end{cases}$ find $\lim\limits_{x\to 1} f(x)$

4. Find the value of 'a' such that $\lim f(x)$ exist, when $f(x)\begin{cases} a+5 & , x < 2 \\ x-1 & , x \ge 2 \end{cases}$

5. Evaluate :

(a) $\lim\limits_{x\to 0} \frac{e^{2x}-1}{x}$

(b)

(c) $\lim\limits_{x\to 0} \frac{\sin 4x}{2x}$

(d) $\lim\limits_{x\to 0} \frac{\sin ax}{\sin bx}$

(e) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$

[Overlapping expressions printed on top of one another; best readings of the parts:]

$\lim\limits_{x\to 0} \frac{1-\cos 8x}{x}$ $\lim\limits_{x\to 0} \frac{\sin 2x(1-\cos^2 x)}{x^3}$ $\lim\limits_{x\to 0} \frac{e^x + e^{-x} - 2}{x}$ $\lim\limits_{x\to 1} \frac{\cos\frac{\pi x}{2}}{1-x}$

(f)

(g)

(h) $\lim\limits_{x\to 0} \frac{1-\cos^2 x}{3\tan^2 x}$

(i) $\lim\limits_{x\to \pi} \frac{\sin x}{\pi - x}$

(j)

(k) $\lim_{\theta \to 0} \frac{\tan 7\theta}{\sin 4\theta}$

(l)

6. (a) Show that $f(x) = e^{5x}$ is a continuous function.

(b) Show that $f(x) = e^{-2x+5}$ is a continuous function.

7. (a) If $f(x) = 2x + 1$, when $x \neq 1$ and $f(x) = 3$ when $x = 1$ show that the function $f(x)$ is continuous at $x = 1$.

(b) If $f(x) = \begin{cases} 4x+3 & , x \neq 2 \\ 3x+5 & , x = 2 \end{cases}$, find whether the function f is continuous at $x = 2$.

(c) Examine the continuity of $f(x) = |x-2|$ at $x = 2$.

8. For what value of k is the following function continuous at $x = 1$?

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & , \text{when } x \neq 2 \\ k & , \text{when } x = 1 \end{cases},$$

$$\lim_{\theta \to 0} \frac{\tan 7x + \tan 5x}{4x} \qquad f(x) = \frac{x^2+2x+6}{x^2+8x-16} \qquad f(x) = \frac{\ldots}{(x-2)(x-3)}$$

9. At would points is the functions f (x) continuous in each of the following cases?

(a) $f(x) = \frac{x-3}{(x-1)(x-4)}$

(b)

(c)

(d)

MISCELLANEOUS QUESTION

Part-B

1. Evaluate :

(a)

(b) $\lim_{x\to 2} \frac{x^2+2x}{x^2+x^2-2x}$

(c)

(d) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

(e)

(f)

[illegible]

2. Find the left hand limit and right hand limit of the following questions : $f(x)=\frac{x^2-1}{|x-1|}$ as $x\to 1$

3. Evaluate :

(a) $\lim_{x\to 0}\left[\frac{e^x+e^{-x}-2}{x^2}\right]$

(b)

4. Examine the continuity of the following :

$$f(x)\begin{cases}\frac{1}{x}-x & 0<x<\frac{1}{2}\\ \frac{1}{2} & x=\frac{1}{2}\\ \frac{3}{2}-x & \frac{1}{2}<x<1;\text{at } x=\frac{1}{2}\end{cases}$$

5. Determine the point of discontinuity, if the following functions :

(a) $\dfrac{x^2+3}{x^2+x+1}$ (b)

(c) (d)

4xx22++x3+xx1+4 5 − 16   x ≠ 2

xx22 −+32xx++11   16   x = 2

$$f(x)=\begin{cases}\dfrac{x^4-16}{x-2} & x\neq 2\\ 16 & x=2\end{cases}$$

Chapter - 5

CONCEPT OF DIFFERENTIATION

DERIVATIVES :

Suppose f is a real valued function and a is a point in its domain of definition. The derivative of *f* at a is defined by $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Provided this limit exists. Derivative of f(x) at a demated by $f^1(a)$.

For Example : Find derivative at x = 2 of the function f(x) = 3x

Sol. We have

$$f^1(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{3(2+h)-3\times 2}{h}$$

$$=\lim_{h\to 0}\frac{6+3h-6}{h}$$

$$=\lim_{h\to 0}3$$

$$=3$$

In other words

Suppose f is real valued function the function defined by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Wherever the limit exists is defined to be the derivative of *f* at x and is denoted by f(x). This definition of derivative is also called the first principle of derivative.

Thus

$$\frac{d}{dx}f(x)=f^1(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

For example : Find derivative of X^2 by first principle.

Sol. Let $f(x) = x^2$ .....(1)

and $f(x+h) = (x+h)^2$ .....(2)

We know that $\frac{d}{dx} f(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$ ....(3)

Now from equation (1), (2) and (3) we get

$$f'(x) = \lim_{x \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{x \to 0} \frac{x^2 + h^2 + 2hx - x^2}{h}$$

$$= \lim_{x \to 0} \frac{h(h+2x)}{h}$$

$$= \lim_{x \to 0} (h+2x)$$

$= 2x$

Example : Find derivation of sin x by first principle.

Sol. Let $f(x) = \sin x$ ....(1)

and $f(x+h) = \sin(x+h)$ ....(2)

We know that :

$$\frac{d}{dx} f(x) = f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h} \quad ....(3)$$

Using equation (1), (2) and (3)

$$f'(x) = \lim_{x \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{x \to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right).\sin\left(\frac{x+h-x}{2}\right)}{h} \quad \left[\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)\right]$$

$$= \lim_{x \to 0} \frac{2\cos\left(x + \frac{h}{2}\right).\sin\frac{h}{2}}{h}$$

$$= \lim_{x\to 0} \cos\left(x+\frac{h}{2}\right) . \lim_{\frac{h}{2}\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$= \cos (x + 0) \times 1$

$= \cos x$

ALGEBRA OF DERIVATIVES OF FUNCTIONS :

1. $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

2. $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

3. $\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

4. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$

$\frac{d}{dx}(K.f(x)) = K\frac{d}{dx}f(x)$ $(a^x) = a^x \log_e a$

5. $\frac{d}{dx}(C) = 0$

6.

DIFFERENTIATION OF SOME STANDARD FUNCTIONS :

1. $\frac{d}{dx}(x^n) = n.x^{n-1}$

2. $\frac{d}{dx}(e^x) = e^x$

3.

4. $\frac{d}{dx}(\log e^x) = \frac{1}{x}$

5.

6. $\frac{d}{dx}(\sin x) = \cos x$

7. $\frac{d}{dx}(\cos x) = -\sin x$

8. $\frac{d}{dx}(\tan x) = \sec^2 x$

9. $\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$

10. $\frac{d}{dx}(\sec x) = \sec x.\tan x$

11. $\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x.\cot x$ [illegible]

12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -k\, x < 1$

13. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -k\, x < 1$

14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, -\infty < x < \infty$

15.

16.

MISCELLANEOUS QUESTION

Part-A

1. Find the derivative of each of the following function by delta method :

   (a) $10x$

   (b) $2x + 3$

   (c) $3x^2$

   (d) $x^2 + 5$

2. Findthe velocity of particles moving along a straight line for the given time distance relation t the given values of time t.

   (a) $s = 2 + 3t$; at $t = 2/3$

   (b) $s = 8 + - 7$; at $t = 4$

   (c) $s = 7t^2 - 4t + 1$; at $t = 5/2$

3. Find the derivative of each of the following functions using ab initio method.

   (a) $\frac{1}{x} \quad x \neq 0$

   (b)

   ab + 1 b — x ≠ 0 x ≠ −b/a — −d / c — √ax + b — bx + d

   (c)

   (d) $x + \frac{1}{x} \quad x \neq 0$

4. Find the derivative of each of the following functions from first principles :

   (a)

   (b)

   (c) $\sqrt{x} + \frac{1}{\sqrt{x}}; \ x \neq 0$

   (d)

5. (a) If $f(x) = 20x^9 + 5x$, find $f'(0)$, $f'(3)$, $f'(8)$

(b) If $f(x) = \frac{x^8}{8} - \frac{x^6}{6} + \frac{x^4}{4} - 2$, find $f'(-2)$.

(c) If find at $r = 2$.

(d) If find $f'(0)$, $f'(1)$.

6. Find the derivative of each of the following functions by product rule :

(a) $f(x) = (3x + 1)(2x - 7)$

(b) $y = (2x + 1)(-2x - 9)$

(c) $y = x^2(2x^2 + 3x + 8)$

(d) $4(x) = (x^2 - 4x + 5)(x^3 - 2)$

7. Find the derivative of each of the following functions :

(a) $f(r) = r(1 - r)(\pi r^2 + r)$

(b) $f(x) = (x - 1)(x - 3)(x - 5)$

(c) $f(x) = (3x^2 + 7)(5x - 1)(3x^2 + 9x + 8)$ [illegible]

(d) $y = \frac{2}{5x - 7}$

(e)

(f)

(g)

(h)

(i)

(j) $f(x) = \dfrac{x(x^2+3)}{x-2}$

(k) $y = \dfrac{1}{\sqrt{7-3x^2}}$

(l) $y = \sqrt[3]{(x^2+1)^5}$

(m) $y = (2x^2 + 5x - 3)^{-4}$

(n) $y = \left[\dfrac{1}{6}x^6 + \dfrac{1}{2}x^4 + \dfrac{1}{16}\right]^5$

(o)

8. Find the derivative of second order of the following functions :

(a) x^3

(b) $x^4 + 3x^2 + 9x^2 + 10x + 1$

(c) [illegible] $\cdot\sqrt{x^2+8}$

(d)

MISCELLANEOUS QUESTION

Part-B

1. The distance 5 meters travelled in time t seconds by a case is given by the relation $s = t^2$. Calculate :

(a) The rate of change of distance with respect to time (t).

(b) The speed of car at time t = 3 sec.

2. Given $f(t) = 3 - 4t^2$, use delta method to find f' (t) f' (1/3).

3. Find the derivate $f(x) = x^4$ from the first principles. Hence find f' (0),

4. Find the derivative of the function from the first principles.

5. Find the derivatives of the function by the first principles :

(a) ax + b, where a and b are constants.

(b) $2x^2 + 5$

(c) $x^3 + 3x^2 + 5$

(d) $(x - 1)^2$

6. Find the derivative of each of the following functions :

(a) $f(x) = px^4 + qx^2 + 7x - 11$

(b) $f(x) = x^3 - 3x^2 + 5x - 8$

(c)

(d)

7. Find the derivative of each of the functions given below by two ways, first by product rule and then by expanding the product. Verify the two answers are same :

(a) $y = \sqrt{x}\left(1 + \frac{1}{\sqrt{x}}\right)$

(b)

8. Find the derivative of the following functions : [illegible]

(a)

(b)

(c) $f(x) = \frac{1}{1 + x^4}$

(d)

(e) $f(x) = \frac{x - 4}{2\sqrt{x}}$

(f) $f(x) = \frac{3x^2 + 4x - 5}{x}$

(g) $f(x) = \dfrac{(x^3+1)(x-2)}{x^2}$

9. Use chain rule, to find the derivative of each of the function given below :

(a) $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$

(b)

(c)

10. Find the derivatives of second order of each of the following :

(a)

(b)

(c) $(x^2 + 1)(x - 1)$

(d)

[illegible]

MISCELLANEOUS QUESTION

Part-C

1. If find $\dfrac{dy}{dx}$.

2. Evaluate, at $x = \dfrac{\pi}{2}$ and 0.

3. If $y = \dfrac{5x}{\sqrt[3]{(1-x)^2}} + \cos^2(2x+1)$; find $\dfrac{dy}{dx}$.

4. If $y = \sec^{-1}\dfrac{\sqrt{x+1}}{\sqrt{x+1}} + \sin^{-1}\dfrac{\sqrt{x-1}}{x+1}$, then show that $\dfrac{dy}{dx} = 0$.

5. If $x = a\cos^3 x$, $y = a\sin^3 x$, then find

6. If find $\frac{dy}{dx}$.

7. Find the derivative of $\sin^{-1} x$ with respect to

8. If y = cos (cos x), prove that $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} + y \sin^2 x = 0$.

9. If $y = \tan^{-1} x$ show that $(1 + x^2) y_2 + 2xy_1 = 0$

10. If $y = (\cos^{-1} x)^2$. Show that $(1 - x^2) y_2 - xy_1 - 2 = 0$.

11. Find the derivative of with respect to x by first principle.

12. Find the derivative of the each of the following :

(a) $\sin^{-1} \sqrt{x}$

(b) $\cos^{-1} x^2$

(c) $\tan^{-1} \frac{\cos x}{1 + \sin x}$

(d) $(2 \sin^{-1} x)$

13. Find $\frac{dy}{dx}$ if [illegible]

MISCELLANEOUS QUESTION

Part-D

1. Find the derivative of each of the following functions :

(a) $(x^x)^x$

(b) $(x)^{x^x}$

2. Find if :

(a) $y = a^{x \log \sin x}$

(b) $y = (\sin x) \cos^{-1} x$

(c)

(d)

3. Find the derivative of each of the functions given below :

(a)

(b) $f(x) = \sin^{-1} x.\ x^{\sin x}.e^{2x}$

4. Find the derivative of each of the following functions :

(a) $y = (\tan x)^{\log x} + (\cos x)^{\sin x}$

(b) $y = x^{\tan x} + (\sin x)^{\cos x}$

5. Find $\frac{dy}{dx}$:

(a)

(b) $y = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

6. Find $\frac{dy}{dx}$, if :

$f(x) = x \cos x . \log x . e^{x^2} . x^x$ $y = \frac{2x^4 \sqrt{\cot x\ 6}}{(3x\sqrt{x} + 5)^2}$

(a) $y = a^x.x^a$

(b) $y = 7x^2 + 2x.$

7. Find the derivative of the following functions :

(a) $y = x^2\, e^{2x} \cos 3x$

(b)

8. If $y = x^{x^{x^{x\ldots\ldots\infty}}}$ prove that $x\frac{dy}{dx} = \frac{y^2}{1 - y \log x}$

9. Find the second order derivative of each of the following :

(a) e^x

(b) $\cos (\log x)$

(c) x^x

Chapter - 6
APPLICATIONS OF DERIVATIVES

RATE OF CHARGE OF QUANTITIES :

One quantity of varies with another quantity x satisfying some rule y = f(x), then $\frac{dy}{dx}$ (or f' (x) represent the rate of change y with respect to x and (or f'(x)) represent the rate of change of y with respect to x at $x = x_0$

For there, if two variables x any are varying with respect to another variable t i.e. if x = f(t) and y = g (t) then by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}, \text{if } \frac{dx}{dt} \neq 0$$

Thus the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to.

$\left(\frac{dy}{dx}\right)_{x=x_0}$ $\frac{3}{2}(2x+3)$ diameter $= \frac{3}{4}(2x+3)$

For example : 1. A balloon which always remains spherical, has a variable diameter determine the rate of change of volume with respect to x.

Sol. Let r be the radius of spherical balloon and volume is v.

diameter of spherical balloon =

radius

$$\therefore \ r = \frac{1}{2} \times \frac{3}{2}(2x+3)$$

r

$\therefore$ Volume of spherical balloon

(V)

V

V

Diff. this w.r. to x we get

Hence volume of spherical ballon is changing at the rate of $\frac{27}{8}\pi(2x+3)^2$ unit³/unit.

Example 2 : The total revenue received from the sale of x units of a product is given by

$$R(x) = 10x^2 + 13x + 24$$

Find marginal revenue when x = 5, where by marginal revenue we mean the rate of change of total revenue w.e.f. to the number of items sold at an instant.

Sol. Given $R(x) = 10x^2 + 13x + 24$

Since marginal revenue is the rate of change of the revenue with respect to the number of items sold.

$\therefore$ Marginal revenue $(MR) = \frac{dR}{dx} = 20x + 13$

When x = 5, MR = 20 × 5 + 13

MR = 113 Rs.

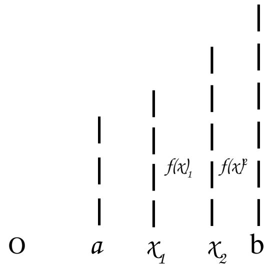
STRICTLY INCREASING FUNCTION :

A function f(x) is said to be a strictly Increasing function on (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad \text{for all } x_1, x_2 \in (a, b)$$

Thus f(x) is strictly increasing on (a, b) if the values of f(x) in crease with the increase in the values of x. Graphically, f(x) is increasing on (a, b) if the graph y = f(x) moves up as x moves to the right. The graph of strictly Increasing function as shown below.

STRICTLY INCREASINGH FUNCTION

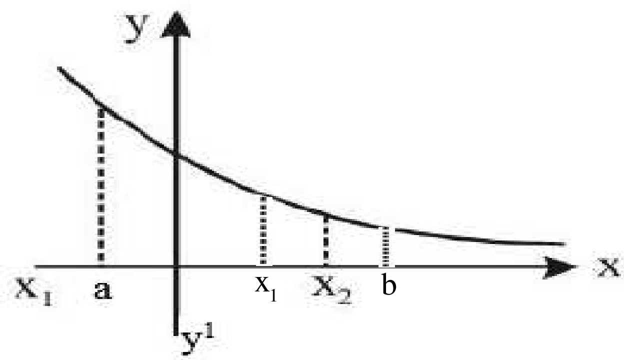


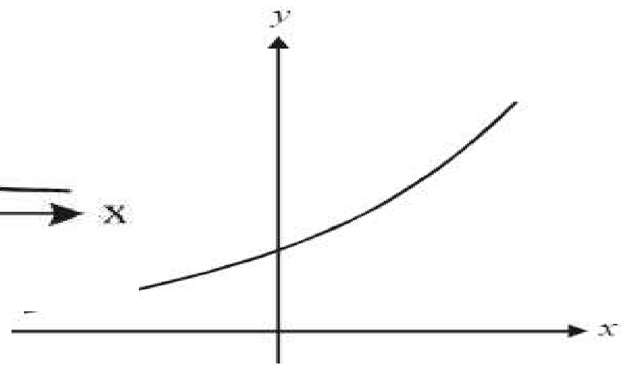
STRICTLY DECREASING FUNCTION :

A function f(x) is said to be a strictly decreasing function on (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in (a, b)$$

Thus f(x) is strictly decreasing on (a, b) the values of f(x) decrease with the increase in the values of x. Graphically it means that f(x) is a decreasing function on (a, b) it its graph moves down as x moves to the right. The graph of strictly decreasing function as shown below :

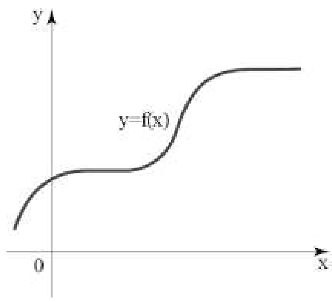




INCREASING FUNCTION :

A function f(x) is said to be increasing function on (a, b) if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in (a, b)$

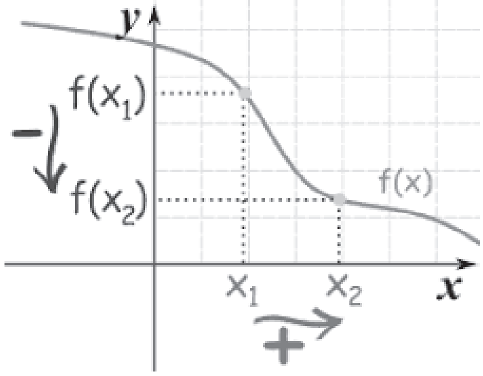
Thus f(x) is increasing on (a, b) when the values of f(x) increase and constant at a time with the increase in the values of x. The graph increasing function as shown below.



DECREASING FUNCTION :

A function f(x) is said to be decreasing function on (a, b) if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in (a, b)$

Thus f(x) is decreasing on (a, b) when the values f(x) decreases as well as constant at a time with increase in the vales of x. The graph of decreasing function as shown below.



TANGENTS AND NORMALS

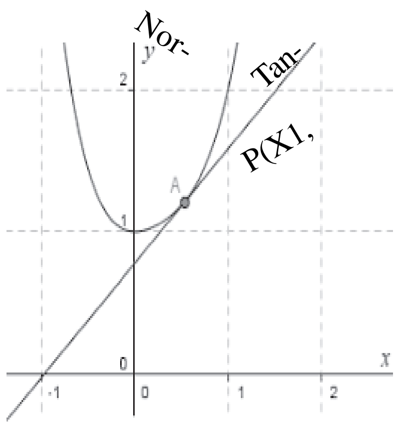
SLOPE OF THE TANGENT :

Let y = f(x) be a continuous wand let $p(x_1, y_1)$ be a point on its. Then at point p is slope of the tangent to the curve y = f(x) at point .

$$\frac{dy}{dx}\text{atp} = \left(\frac{dy}{dx}\right)_{\text{at p}} = \tan\alpha$$

i.e. slope of tangent at

where α is angle which the tangent at P makes with the positive direction of x-axis.



Slope of The Normal : The normal to a curve at $P(x_1, y_1)$ is a line perpendicular to the tangent p and passing through.

$\therefore$ Slope of normal at p

$$= -1 \text{ slope of tangent at p}$$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_p}$$

EQUATIONS OF TANGENT AND NORMAL :

We know that equation of a line passing through a point (x_1, y_1) and having slope is

$$y - y_1 = m\,(x - x_1)$$

Therefore the equation of the tangent at $P\,(x_1, y_1)$ to the curve $y = f(x)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_p (x - x_1)$$

and the equation of Normal at $P\,(x_1, y_1)$ to the curve $y = f(x)$ ios

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_p}(x - x_1)$$

APPROXIMATIONS :

Let $y = f(x)$ be a function of x and Δx be a small change in x and let Δy be the corresponding change in y then

$$\lim_{\Delta t \to 0} \frac{\Delta t}{\Delta x} = \frac{dy}{dx} = f'(x)$$

$$\Rightarrow \qquad \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \in \text{ where} \in \to 0 \text{ as } \Delta x \to 0$$

$$\Rightarrow \qquad \Delta y = \frac{dy}{dx} \Delta x + \in \Delta x$$

is a very-very small quantity that can be neglected

$\therefore \; \Delta y$ approximately

and also $\Delta y = f(x + \Delta x) \ldots\ldots f(x)$

For Example 1 : Find the approximate value of f (3.02) where $f(x) = 3x^2 \; 5x + 3$

Sol. Let x = 3 and $x + \Delta x = 3.02$ then $\Delta x = 0.02$

We have $f(x) = 3x^2 + 5x + 3$

When x = 3

$f(3) = 3(3^2) + 5 \times 3 + 3$

$= 3 \times 9 + 15 + 3 = 45$

$$\therefore \Delta y = \frac{dy}{dx} \Delta x \qquad \frac{\Delta x}{x} \times 100 = 2 \qquad \Delta x = \frac{2x}{100}$$

$= (6x + 5) \Delta x$

$\Delta y = (6 \times 3 + 5) \times 0.02$

$\Delta y = 0.46$

$\therefore f(3.02) = y + \Delta y = 45 + 0.46$

$= 45.46$

Hence approximate value of f (3.02) is 45.46.

Example 2. Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by $2y_1$

Sol. Let Δx be the change in x and Δv by the corresponding change in v.

Given that

$\Rightarrow$

$\therefore \; v = x^3$

$$\frac{dv}{dx} = 3x^2$$

$$\Delta v = \frac{dv}{dx} \Delta x$$

$$\Delta v = \frac{6}{100} v$$

Hence, the approximate change in volume is 6%.

MAXIMA AND MINIMA

MAXIMUM :

Let f(x) be a function with domain DCr. Then f (x) is said to attain the minimum value at a point a $\in$ D if f (x) = f(a) for all x $\in$ D.

In such a case, the point a is called the point of minima and f(a) is known as the minimum value or the least value or the absolute minimum value of f(x).

$$\Delta v = 3x^2 \times \frac{2x}{100}$$

LOCAL MAXIMUM :

A function f(x) is said to attain a local maximum at x = a if there exists a neighbourhood $(a - \delta, a + \delta)$ of a such that

$f(x) < f(a)$ for all $x \in (a - \delta, a + \delta), x \neq a$

or $f(x) - f(a) < 0$ for all $x \in (a - \delta, a + \delta), x \neq a$

In such a case f(a) is called the local minimum value of f(x) at x = a.

LOCAL MINIMUM :

A function f (x) is said to attain a local minimum at x = a if there exists a neighbourhood $(a - \delta, a + \delta)$ of a such that

$f(x) > f(a)$ for all $x \in (a - \delta, a + \delta), x \neq a$

or $f(x) - f(a) > 0$ for all $x \in (a - \delta, a + \delta), x \neq a$

In such a case f (a) is called the local minimum value of f (x) at x = a.

First Derivative Test for Local Maxima and Minima

Let f(x) be a function differentiable at x = a then

(A) x = a is a point of local maximum of f(x) if

(i) f'(a) = 0 and

(ii) f'(x) changes sign from positive to negative as x passes through a *i.e.* f'(x) > 0 at every point in the left neighbourhood (a – δ, a), of a and f'(x) < 0 at every point in the right neighbourhood (a, a + δ) of a.

(B) x = a is a point of local minimum of f(x) if

(i) f'(a) = 0 and

(ii) f'(x) changes sign from negative to positive as x passes through a i.e. f'(x) < 0 at every point in the left neighbourhood (a – δ, a) of a f'(x) > 0 at every point in the right nbd (a, a + δ∈) of a.

(C) If f'(a) = 0 but f'(x) does not change sign i.e. f'(a) has the same sign in the complete neighbourhood of a then a is neither a point of local maximum nor a point of local minimum.

SECOND DERIVATIVE TEST :

Let f (x) be a function defined on an interval I and C∈ I left be twice differentiable at C. Then

(i) x = c is a point of local maxima f'(c) = 0 and f''(c) < 0

Then f(c) is local maximum value of f(x).

(ii) x = C is a point of local minima

If f'(c) = 0 and f''(c) > 0

Then f(c) is local minimum f(x).

(iii) The test fail if f'(c) = 0 and f''(c) = 0. In this case, we go back to the first derivative and find whether C is a point of local maxima, local minima or point inflexion.

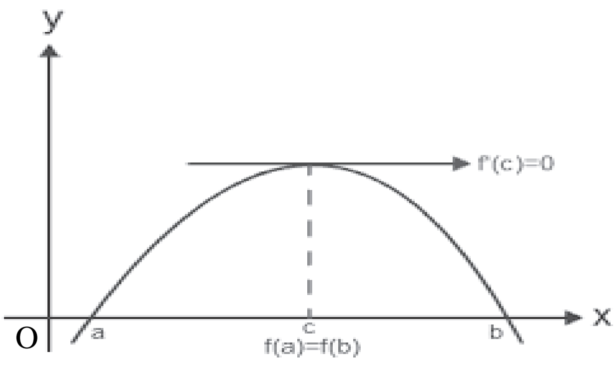
ROLLE'S THEOREM :

Let f be a real function defined in the closed interval [a, b] such that :

(i) f is continuous in the closed interval [a, b]

(ii) f is differentiable in the open interval (a, b)

(iii) f(a) = f(b)



There is at least one point C in the open interval (a, b) such that f'(c) = 0

Example : Verify Rolle's theorem for the function f'(x) = (x – 1), (x – 2), x ∈ [0, 2]

Sol. f(x) = x (x – 1) (x – 2)

$f(x) = x^3 - 3x^2 + 2x$

(i) f (x) is a polynomial function and hence continuous in [0, 2]

(ii) f(x) is differentiable on [0, 2]

(iii) Also f(0) = 0 and f(2) = 0

∴ f(0) = f(2)

All the conditions of Rolle's theorem are satisfied.

Also $f'(x) = 3x^2 - 6x + 2$

∴ f'(c) = 0 gives

$3C^2 - 6C + 2 = 0$

$$\Rightarrow \quad C = \frac{6 \pm \sqrt{36-24}}{6}$$

$$\Rightarrow \quad C = 1 \pm \frac{1}{\sqrt{3}}$$

We see that both the values of C in (0, 2)

LANGRANGE'S MEAN VALUE THEOREM :

Let f be a real value function defined on the closed interval [a, b] such that :

(a) f is continuous on [a, b] and

(b) f is differentiable in (a, b)

(c) f (b) ≠ f(a)

then there exists a point C in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

MISCELLANEOUS QUESTIONS

Part A

1. Find the rate of change of area of a circle with respect to its variable radius r, when r = 3cm.
2. A balloon which always remains spherical, has a variable diameter $\frac{3}{2}(2x+3)$. Determine the rate of change of volume with respect to x.
3. A ballon which always ramains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the ballon is increasing, when its radius is 15 cm.

4. A laddery 5m long is leaning against a wall. The foot of the ladder is pulled along the ground, away from the wall, at the rate of 2 mi/sec. How fast is its height on the wall decresing when the foot of ladder is 4m away from the wall.
5. The total revenue received from the sale of x units of a preoduct is given by $R(x) = k10x^2 + 13x + 24$. Find the marginal revenue when x =5, where by marignal revenue with respect to the number of Items sold at an instant.
6. The total cost associated with the production of x units of an Item is given by
(x) $0.007 x^3 - 0.003 x^2 + 15 x + 4000$.
Find the marginal cost when 17 units are produced, where by marginal cost we mean the instantaneous rate of change of the total cost at any level of output.
7. Using differential, Find the approximate value of .
8. Using differentials, Find the approximate value of .
9. Find the approximate value of f (3.02) where $fx = 3x^2 + 5x + 3$.
10. If the radius of a sphere is measured by 9cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.
11. Find the approx. Change in the volume V of a cube of side x meters caused by Incresing the side by 2%.
12. Find the slope of tangent and normal to the curve,
$x^3 + x^2 + 3xy + y^2 = 5$ at (1, 1).
13. Show that the tangents to the curve [illegible] at the point $x = \pm 3$ are parallel.
14. The slope of the curve $6y^3 = px^2 + q$ at (2, –2) is . Find the values of p and q.
15. Find the equation of the tangent and normal to the circle $x^2 + y^2 = 25$ at the point (4, 3).
16. Find the equation of the tangent and normal to the curve $16x^2 + 9y^2 = 144$ at the point)1, 1) where $y_1 > 0$ and $x_2 = 2$.
17. Find the points on the curve at which the tangents are parallel to x-axis.
18. Find the equation of all lines having slope-4 that are tangents to the curve .
19. Find the equation of the normal to the curve $y = x^3$ at (2, 8)
20. Verify Rolle's for the function.
.
21. Discuss the applicability of Rolle's Theorem for f (x) = sin x – sin 2x, x ∈ e [0, π] is a sine function, it is continuous and differentiable on (2, π).
22. Verify Langrange's Mean value theorem for f (x) = (x – 3) (x – 6) (x – 9) on [3, 5].

23. Find a point on the parabola $y = (x - y)^2$ where the tangent is parallel to the chord joining (4, 0) and (5, 1).

24. Prove that the function $f(x) = (4x + 7)$ is monotonic for all values of $x \in R$.

25. Show that $f(x) = x^2$, is a strictly decreasing function for all $x < 0$.

26. Find for what values of x, the function :

$f(x) = x^2 - 6x + 8$.

27. Find the internal in which $f(x) = 2x^3 - 3x^2 - 12x + 6$ is increasing or decreasing.

28. Determine the intervals for which the function $f(x) = \frac{x}{x^2 + 1}$ is increasing or decreasing.

29. Show that :

(a) $f(x) = \cos x$ is decreasing in the interval $0 \leq x \leq \pi$.

(b) $f(x) = x - \cos x$ is increasing for all x.

30. Find the maximum (local maximum) and minimum (local minimum) points of the function $f(x) = x^3 - 3x^2 = 9x$.

31. Find the local maximum and local minimum of the function $f(x) = x^2 - 4x$.

32. Find all local maxima and local minima of the function $f(x) = 2x^3 - 3x^2 - 12x + 8$.

33. Find the local maximum and local minimum of the following function

34. Find the local maximum and local minimum, if any for the function $f(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2}$

35. Find the local minimum of the following function :

$2x^3 - 21x^2 + 36x - 20$

36. Find the local maxima and minima (if any) for the function $f(x) = \cos 4x$:

37. Find the maximum value of $2x^3 - 24x + 107$ in the interval [–3, –1].

38. Find the maximum and minimum value of the function $f(x) = \sin x\,(1 + \cos x)$ in $(0, \pi)$.

39. Find two positive real number whose sum is 70 and their product is maximum.

40. Show that among rectangles of given area, the square has the least perimeter.

41. An open box with a square base is to be made out of a given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$

42. Show that of all rectangles inscribed in a given circle, the square has the maximum area.